

CHAIN HOMOGENEOUS SOUSLIN ALGEBRAS

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ABSTRACT. Assuming Jensen’s principle \diamond^+ we construct Souslin algebras all of whose maximal chains are pairwise isomorphic as total orders, thereby answering questions of Koppelberg and Todorćević.

INTRODUCTION

Souslin’s problem was published in the first issue of *Fundamenta Mathematicae* (1920, p.223):

Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu’un élément) n’empiétant les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?

(A (linearly) ordered set without jumps nor gaps and such that each set of its intervals (containing more than one point) the ones not overlapping onto the others is at most countable, is it necessarily an (ordinary) linear continuum?)

During the following decades, though still far from being solved, Souslin’s problem was reconsidered by a number of mathematicians who established several equivalent formulations in terms of trees ([Kur35, Mil43]), metric spaces ([Kur36]) or Boolean algebras ([Mah48]). When in the 1960’s the independence of ZFC and Souslin’s hypothesis SH, which states that the answer to Souslin’s question is “yes”, was established, general interest was focussed on the techniques developed for such independence results, yet there were also some publications exploring the structural properties of Souslin lines ([Jen69]) and of Souslin trees and algebras ([Jec72a, Jec72b]).

Souslin lines can be analyzed through their partition trees, which are normal Souslin trees, i.e. normal trees of height ω_1 without uncountable antichains. There is also an inverse procedure, which generates a Souslin line from a given Souslin tree. Yet these operations are highly non-canonical, meaning that one Souslin tree can give rise to up to 2^{\aleph_1} non-isomorphic Souslin lines and (in a restricted sense) vice versa. In the present article we give an example of a Souslin tree to which only one complete and dense linear order type is associated. Moreover, this Souslin tree and

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its associated Souslin algebra have interesting homogeneity properties. And this is where we meet a second, algebraic motivation for the research presented here.

Homogeneity of a mathematical object generally means that the object locally looks similar, i.e., that sufficiently small parts of the object are all structurally equal to each other. E.g., for a Boolean algebra B homogeneity is defined as the property that the non-trivial relative algebras $B \restriction a$ for all $a \in B^+$ are isomorphic to each other. Of course, many popular structures satisfy stronger homogeneity conditions than ordinary homogeneity, as e.g., $(\mathbb{R}, <)$, the linear order of the real numbers has for each pair of countable, dense subsets an auto-homeomorphism that maps one countable, dense subset onto the other.

The variant of homogeneity studied in the present text is *chain homogeneity for Boolean algebras*, the property of a Boolean algebra B that for any two maximal chains K_0 and K_1 of B (i.e. subsets of B which are maximal being linearly ordered by the partial order of B) there is an isomorphism (of linear orders) between K_0 and K_1 . Note, that such an isomorphism of linear orders does not necessarily extend to an automorphism of B .

In the case of σ -complete, atomless Boolean algebras it is easy to see that chain homogeneity implies the c.c.c., so the only possible order types for maximal chains of such algebras are the real unit interval $[0, 1]$ or a Souslin line with endpoints. The only known, complete, atomless and chain homogeneous Boolean algebras are the Cohen-algebras and measure algebras, both having only separable maximal chains.

The present article gives the affirmative answer to the question of Sabine Koppelberg whether there are, under appropriate assumptions, complete and atomless chain homogeneous Boolean algebras with inseparable maximal chains. It is a completely revised version of Chapter 2 of the authors PhD thesis [SF08].

The organisation of the text is quite conventional. In Section 1 we review the basic definitions concerning Souslin lines, trees and algebras and fix the notation. The following three sections introduce the technology we utilize in our Souslin tree constructions, which consists of relations and mappings on the tree under construction and results on the extendibility thereof when adding a new level to the tree. In Section 5 the main results are stated, the proof of which, a somewhat involved Souslin tree construction, is prepared in Section 6 and carried out in Section 7. The paper closes with a section containing some further remarks and open problems.

1. PRELIMINARIES

We collect the basic definitions and results and fix the notation and for later reference. Most proofs are fairly straight forward and therefore omitted, yet references are given. The ambient theory is ZFC, Zermelo-Fraenkel set theory with choice; further hypotheses are explicitly stated.

1.1. Boolean algebras. Boolean algebras and the most elementary related notions, such as subalgebras, completeness, atoms will not be defined here, though we will mention some more or less subtle relationships between them. The main reference for looking up undefined notions is the first volume of the *Handbook of Boolean Algebras*, [Kop89].

Let B be a Boolean algebra. We call the binary relation on B

$$x \leq_B y \iff xy = x \iff x - y = 0$$

the *natural* or the *canonical order* of B . A subset $X \subset B$, which is totally ordered by \leq_B is a *chain* of B . A chain X is a *maximal chain* of B if X is furthermore \subset -maximal amongst the chains of B . We call a Boolean algebra all of whose maximal chains are pairwise order isomorphic *chain homogeneous*. We let

$$\text{mc } B = \{K \subset B \mid K \text{ is a maximal chain of } B\}.$$

An *antichain* of B is a family $X \subset B$ of pairwise disjoint (i.e. $xy = 0$) elements of B . We say that B satisfies (or has) the *countable chain condition* (or short *c.c.c.*) if every antichain of B is at most countable.

Lemma 1.1. *If B is σ -complete and chain homogeneous, then B satisfies the countable chain condition.*

Proof. Given an uncountable antichain X of B it is easy to construct well-ordered chains with supremum 1, $K_0 = \{x_0 <_B x_1 <_B \dots\}$ of order type ω and $K_1 = \{y_0 <_B y_1 <_B \dots\}$ of order type ω_1 . Now for any pair $K, K' \in \text{mc } B$ with $K_0 \subset K$ and $K_1 \subset K'$ there should be an isomorphism $\varphi : K \rightarrow K'$ but then $n \mapsto \min\{\alpha \mid \varphi(x_n) \leq_B y_\alpha\}$ would give a cofinal countable sequence in ω_1 . \square

For any subset X of a complete Boolean algebra B we call

$$\langle X \rangle^{\text{cm}} := \bigcap \{A \mid X \subseteq A \text{ and } A \text{ is a complete subalgebra of } B\}$$

the *subalgebra of B that is completely generated by X* and maybe write $\langle X \rangle_B^{\text{cm}}$ if B is not clear from the context. If the superscript “cm” is omitted, i.e., by $\langle X \rangle_B$, we denote the intersection of all subalgebras containing X as a subset (and not only the complete subalgebras) and call $\langle X \rangle$ the subalgebra of B , that is *finitarily* generated by X .

Note that given an arbitrary Boolean algebra A which is a subalgebra of a complete Boolean algebra B , the (*Dedekind*) *completion* \overline{A} (cf. [Kop89, Section 4.3]), which is the unique complete Boolean algebra containing A as a dense subalgebra, is not necessarily isomorphic to $\langle A \rangle_B^{\text{cm}}$. A *regular subalgebra* A of B is a subalgebra, such that for all $M \subset A$ we have $\sum^B M = \sum^A M$ if the latter sum exists. (This definition

also extends to *regular sublattices*.) A well-known fact is, that A is a regular subalgebra of B if and only if A is a dense subset of $\langle A \rangle_B^{\text{cm}}$. In this case $\langle A \rangle_B^{\text{cm}}$ is indeed isomorphic to \overline{A} , the Dedekind completion of A (cf. [Kop93, Prop.4]).

A Boolean algebra B is \aleph_0 -*distributive* if for every family $(a_{ij})_{i \in \omega, j \in J}$ with an index set J of arbitrary size, the following equation holds:

$$\prod_{i \in \omega} \sum_{j \in J} a_{ij} = \sum \left\{ \prod_{i \in \omega} a_{if(i)} \mid f \in {}^\omega J \right\}.$$

We do not need more specific concepts of distributivity.

We frequently consider the *regular open algebra* $\text{RO } X$ of some topological space X . A subset U of X is regular open if the interior of the closure of U is equal to U and $\text{RO } X$ is the set of all regular open subsets of X . The regular open algebra of any space X is a complete Boolean algebra but it is in general not a subalgebra of $\mathcal{P}(X)$ from which the operations are modified by taking the regularisations (cf. [Kop89, Theorem 1.37]). If the topology of X is not specified, then one of following applies:

- X is a linear order; then X carries the order topology (as in the following proposition),
- $X = T$ is a tree; then T carries the partial order topology which is generated by all subsets of the form $T(t) := \{s \in T \mid t \leq_T s\}$ for some $t \in T$ (a declaration of our tree notation follows below).

1.2. Normal trees. A *tree* is a well-founded partial order $(T, <_T)$ where the set of predecessors $\{s \mid s <_T t\}$ is totally (and thus well-) ordered by $<_T$ for all $t \in T$. The elements of a tree are called *nodes*. The *height* of the node t in T is the order type of the set of its predecessors under the ordering of T , $\text{ht}_T(t) := \text{ot}(\{s \mid s <_T t\}, <_T)$. For an ordinal α we let T_α denote the set of nodes of T with height α . If $\text{ht}_T(s) > \alpha$ we let $s \upharpoonright \alpha$ be the unique predecessor of s in level T_α .

The *height of a tree* T , $\text{ht } T$, is the minimal ordinal α such that T_α is empty. An *antichain* is a set of pairwise incomparable nodes of T , so for $\alpha < \text{ht } T$, the level T_α is an antichain of T .

If c is a subset of the height of a tree T , let $T \upharpoonright c$ denote the tree that consists of all the nodes of T whose height lies in c together with the inherited tree order $<_T$:

$$T \upharpoonright c = \bigcup_{\alpha \in c} T_\alpha, \quad s <_{T \upharpoonright c} t \iff s <_T t.$$

For a node $t \in T$ we let $\text{succ}(t)$ be the set of t 's immediate successors. By $T(t)$ we denote the set of all successors of t in T (including t) which forms a tree with root t under the ordering inherited from T . Nodes, that do not have $<_T$ -successors, are called *leaves*, and T is called κ -branching, κ a cardinal, if all nodes of T have exactly κ immediate successors, except for the leaves.

A *branch* is a subset b of T that is linearly ordered by $<_T$ and closed downwards, i.e. if $s <_T t \in b$ then $s \in b$. For $\alpha < \text{ot}(b, <_T)$ we let $b \restriction \alpha$ be the unique element of $T_\alpha \cap b$ and so extend the similar notation for nodes and their predecessors to branches and their elements. (For a normal tree this is just natural, since the nodes can be identified with the branches leading to them.) A branch is *maximal* in T if it is not properly contained in any other branch of T .

Under the notion of a *normal* tree we subsume the following four conditions:

- (i) there is a single minimal node called the *root*;
- (ii) each node (except for the top level nodes) has at least two immediate successors;
- (iii) each node has successors in every higher non-empty level;
- (iv) branches of limit length have unique limits (if they are extended in the tree), i.e., if s, t are nodes of T of limit height whose sets of predecessors coincide, then $s = t$.

Whenever we consider a mapping $\varphi : T \rightarrow S$ between trees and call this mapping a *tree homomorphism* we mean that φ carries $<_T$ to $<_S$ and respects the height function: $\text{ht}_S(\varphi(s)) = \text{ht}_T(s)$ for all $s \in T$.

A tree T is said to be *homogeneous*, if for all pairs $s, t \in T$ of the same height there is a tree isomorphism between $T(s)$ and $T(t)$, the trees of nodes in T above s and t respectively.

1.3. Souslin lines, trees and algebras. It is well-known that the existence statements of Souslin algebras, lines and trees are mutually equivalent, yet independent of ZFC: they all exist if and only if Souslin's hypothesis SH fails.

A *Souslin line* is a total order directly witnessing the failure of SH, i.e., it is a complete, dense linear order that satisfies the countable chain condition but is not separable, i.e., it has no countable dense subset, but all families of pairwise disjoint open intervals are countable.

In this definition the possibility of a Souslin line having a separable non-trivial interval is included. We will, however, never consider a Souslin line with a non-trivial separable interval.

It is a standard result, that every Souslin line L has a dense subset of cardinality \aleph_1 and therefore the cardinality $|L| = 2^{\aleph_0}$.

In general, a *Souslin tree* is a tree T of height ω_1 such that every family of pairwise incomparable nodes and every branch of T are at most countable. We will only consider normal Souslin trees, where absence of uncountable antichains already implies that the tree has no cofinal branch.

Souslin lines and Souslin trees are tightly related by the following operations (also cf. [Tod84] or [Jec03, 9.14]). If we are given a Souslin tree T and a total order of its supporting set (or just total orderings of the sets of the form $\text{succ}(t)$ for all $t \in T$) we can equip the set of all maximal branches of T with the lexicographical ordering. The

resulting linear order is dense, c.c.c. and inseparable, i.e., its Dedekind completion is a Souslin line.

Now let L be a given Souslin line (without separable intervals). Then every normal partition tree of L is a Souslin tree. A partition tree of a linear order L is a tree whose nodes are open intervals of L such that the union of each level is dense in L and whose tree ordering is given by the inverted inclusion.

Anyway, in general these operations are highly non-canonical; one Souslin tree may be associated to many non-isomorphic Souslin lines and vice versa. To unify considerations one uses the notion of Souslin algebras.

A *Souslin algebra* is a complete, atomless, c.c.c. and \aleph_0 -distributive Boolean algebra¹. We will use the following non-standard denotations: A Souslin algebra that has a set of complete generators of size \aleph_1 will be called *small*; otherwise it is *big*. Small Souslin algebras always have 2^{\aleph_0} elements, while a famous result of Solovay states that the cardinality of (big) Souslin algebras is at most 2^{\aleph_1} (cf. [Jec03, Thm.30.20]).

We use letters $\mathbb{A}, \mathbb{B}, \mathbb{C}$ to denote Souslin algebras. In a Souslin algebra \mathbb{B} , a complete subalgebra \mathbb{A} is atomic if and only if \mathbb{A} is completely generated by some countable subset X of \mathbb{B} (cf. [Kop89, Prop.14.8]). In the other extreme, since distributivity and the c.c.c. are handed down to complete subalgebras, the atomless, complete subalgebras of a Souslin algebra \mathbb{B} are Souslin algebras as well.

We now review the connections between Souslin algebras and Souslin lines and trees respectively, beginning with the lines.

- Lemma 1.2** ([Kop89, §14, Exrc.5], for a proof cf. [SF08, 2.1.2-5]). *a) The maximal chains of a Souslin algebra \mathbb{B} are Souslin lines with endpoints and without non-trivial separable intervals.*
b) Let L be a Souslin line without separable intervals. The regular open algebra of L is a Souslin algebra. Furthermore, if L has endpoints then $\text{RO } L$ has a maximal chain K which is isomorphic to L such that $\langle K \rangle^{\text{cm}} = \text{RO } L$.
c) Let \mathbb{B} be a Souslin algebra and $K \in \text{mc } \mathbb{B}$. Then $\langle K \rangle$, the subalgebra of \mathbb{B} that is finitarily generated by K , is a regular subalgebra of \mathbb{B} . In particular we have

$$\langle K \rangle^{\text{cm}} \cong \overline{\langle K \rangle} \cong \text{RO } K.$$

It is well known that for a normal Souslin tree T , its regular open algebra $\text{RO } T$ is a Souslin algebra and T can be densely embedded in $\text{RO } T$. On the other hand, in every Souslin algebra \mathbb{B} that has a family of complete generators of cardinality \aleph_1 , i.e., \mathbb{B} is a *small* Souslin algebra, there is a dense subset T of $\mathbb{B} \setminus \{0\}$ such that

¹In other contexts than ours, κ -Souslin algebras are defined as complete, atomless, κ -c.c. and $(< \kappa)$ -distributive Boolean algebras and κ can be any uncountable cardinal. In this notation the objects of our consideration are called \aleph_1 -Souslin algebras. But by Lemma (1.1) above, these higher Souslin algebras for $\kappa > \aleph_1$ always have maximal chains of distinct order types and are therefore never chain homogeneous.

$(T, >_{\mathbb{B}})$ is a normal Souslin tree (note that $<_T$ is $>_{\mathbb{B}}$) that is regularly embedded in \mathbb{B} and whose regular open algebra is therefore isomorphic to \mathbb{B} (see e.g. [Kop89, Thm.14.20]). We will use the following convention established in [DJ74]. Let T be a subset of a Souslin algebra \mathbb{B} . Then T is said to *souslinise* \mathbb{B} or to be a *Souslinisation* of \mathbb{B} if T is dense in \mathbb{B} and becomes a Souslin tree under the reverse Boolean order of \mathbb{B} . Every level T_α of a Souslinisation T of \mathbb{B} is a partition of unity in \mathbb{B} and taking limits in T is simply the evaluation of the corresponding infinite product in \mathbb{B} : If $t \in T_\alpha$ is a limit node, let t_γ for $\gamma < \alpha$ be the unique $>_{\mathbb{B}}$ -predecessor of t of height γ . Then $t = \prod \{t_\gamma \mid \gamma < \alpha\}$. The Souslinisation is unique up to the elimination of a non-stationary set of levels: Given an isomorphism φ between two Souslin algebras with Souslinisations T_1 and T_2 respectively, there is a club $C \subseteq \omega_1$, such that φ becomes an isomorphism of trees when restricted to $T_1 \upharpoonright C$ and $T_2 \upharpoonright C$. For a proof of this well-known result, cf. [DJ74, Lemma VIII.9] or [Jec97, Lemma 25.6].

A Souslinisation T of \mathbb{B} provides a natural stratification of \mathbb{B} by countably generated, complete and therefore atomic subalgebras. Fix a Souslinisation T of \mathbb{B} and let for $\alpha < \omega_1$

$$\mathbb{B}^\alpha := \langle T_\alpha \rangle^{\text{cm}}.$$

Note that for all $\alpha < \omega_1$ we have $\mathbb{B}^\alpha \cong \mathcal{P}(\omega)$ and $T \upharpoonright (\alpha + 1) \subset \mathbb{B}^\alpha$. Clearly the sequence of the \mathbb{B}^α is increasing. To show $\mathbb{B} = \bigcup_{\alpha < \omega_1} \mathbb{B}^\alpha$, pick $a \in \mathbb{B} \setminus \{0\}$. There is a maximal pairwise disjoint subset A of T of elements $\leq_{\mathbb{B}} a$, so $\sum A = a$. By the countable chain condition A must be countable and therefore a subset of \mathbb{B}^α for some $\alpha < \omega_1$. We finally note, that the sequence of the \mathbb{B}^α is not continuous. For a countable limit ordinal α we have

$$\bigcup_{\gamma < \alpha} \mathbb{B}^\gamma \subsetneq \left\langle \bigcup_{\gamma < \alpha} \mathbb{B}^\gamma \right\rangle^{\text{cm}} = \mathbb{B}^\alpha.$$

1.4. Jensen's principle \diamond^+ . As for many Souslin tree constructions in the literature, we will assume diamond-principles.

- Definition 1.3.** a) The sequence $(R_\alpha)_{\alpha < \omega_1}$ of sets $R_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ is a \diamond -sequence if for all $X \subseteq \omega_1$ there is some stationary set $s \subseteq \omega_1$ such that, for all $\alpha \in s$ we have $X \cap \alpha = R_\alpha$. The statement “There is a \diamond -sequence.” will be denoted by \diamond .
- b) The sequence $(S_\alpha)_{\alpha < \omega_1}$ of countable sets $S_\alpha \subseteq \mathcal{P}(\alpha)$ is a \diamond^+ -sequence if for all $X \subseteq \omega_1$ there is some club set $c \subseteq \omega_1$ s.t., for all $\alpha \in c$ we have $X \cap \alpha, c \cap \alpha \in S_\alpha$. The statement “There is a \diamond^+ -sequence.” will be denoted by \diamond^+ .

Since $\diamond^+ \rightarrow \diamond$ we only need to assume \diamond^+ in our statements even though for the sake of convenience we will use both, a \diamond^+ -sequence and a \diamond -sequence, at the same time.

2. GENERATING CHAINS AND WIPERS

For the representation of a given maximal chain K of a Souslin algebra \mathbb{B} we will use the approximations $K_\alpha = K \cap \mathbb{B}^\alpha$ with respect to some fixed Souslinisation T which are chains in \mathbb{B}^α , yet not necessarily maximal ones. In the simple case of power set algebras or their isomorphic copies, such as the \mathbb{B}^α , we can give a nice characterisation of maximal chains.

Proposition 2.1 (folklore, [SF08, 2.1.10]). *Let X be a set. Given a \subset -chain K of $\mathcal{P}(X)$, define the quasi-ordering \leq_K on X by*

$$x \leq_K y \leftrightarrow (\forall u \in K)(y \in u \Rightarrow x \in u)$$

and let $\overline{K} := \{\sum M, \prod M \mid M \subseteq K\}$ be the completion of the linear order (K, \subset) in $\mathcal{P}(X)$. Then the following statements are equivalent:

- (i) \overline{K} is a maximal chain of $\mathcal{P}(X)$;
- (ii) \leq_K is a linear ordering of X ;
- (iii) $\langle K \rangle_{\mathcal{P}(X)}^{\text{cm}} = \mathcal{P}(X)$.

In particular, there is a bijective association between the total orders on X and the maximal chains of $\mathcal{P}(X)$.

Definition 2.2. A maximal chain K of a Souslin algebra \mathbb{B} that completely generates \mathbb{B} will be called a *generating chain*.

We stress that a chain K that satisfies $\langle K \rangle^{\text{cm}} = \mathbb{B}$ will not be called generating unless it is maximal.

Lemma 2.3 (cf. [SF08, 2.2.2]). *Given a generating chain $K \subseteq \mathbb{B}$ the set*

$$C := \{\alpha < \omega_1 : K \cap \mathbb{B}^\alpha \in \text{mc}(\mathbb{B}^\alpha)\} = \{\alpha \in \omega_1 : \langle K \cap \mathbb{B}^\alpha \rangle^{\text{cm}} = \mathbb{B}^\alpha\}$$

is closed and unbounded in ω_1 .

The proof of the last lemma works by a straight forward catch-up argument using the c.c.c. and the distributive law of \mathbb{B} .

Combining Proposition 2.1 and Lemma 2.3, we find a means for the representation of generating chains of a Souslin algebra as a relation on one of its Souslinizations that we now fix in the notion of “wiper”. Requirement (ii) below is of technical importance when it comes to the construction of isomorphisms between wipers.

Definition 2.4. a) Let T be a normal tree with countable levels of arbitrary height $\alpha \leq \omega_1$ and let C be a subset of α . A *wiper of total orders* (or more convenient a *wiper*) on $T|C$ is a family $W = \langle <_\gamma \mid \gamma \in C \rangle$ of total orders $<_\gamma$ on T_γ such that

- (i) W respects the tree order of T : for all $\beta, \gamma \in C$, $\beta < \gamma$ and $s, t \in T_\beta$, $s', t' \in T_\gamma$ we have

$$s <_T s' \wedge t <_T t' \wedge s <_\beta t \Rightarrow s' <_\gamma t',$$

- (ii) if $\beta, \gamma \in C$, $\beta < \gamma$, and $s \in T_\beta$, then the set $I_{s,\gamma} = \{t \in T_\gamma \mid s <_T t\}$ of successors of s in level γ is ordered densely without endpoints by $<_\gamma$.
- b) If T of height ω_1 is a Souslinisation of \mathbb{B} and W is a wiper on T , we say that

$$K_W := \left\{ \sum M \mid (\exists \alpha \in \omega_1) M \subseteq T_\alpha, M \text{ is an initial segment of } <_\alpha \right\}$$

is the subset of \mathbb{B} which is *induced* by W .

To illustrate the definition and justify its definiendum imagine the Souslin tree $T \restriction C$ printed on the windscreen of your car, the levels on concentric lines around some point at the bottom –where the root of T is situated and the windscreen wiper is fixed– and if $s, t \in T \restriction C$ are of height α then s stands to the left of t if and only if $s <_\alpha t$ according to $<_\alpha \in W$. In this picture each position of a windscreen wiper, which has its axis fixed in the root of T , corresponds to a member a of the generating chain K_W induced by the wiper W . And this member $a \in K_W$ can be calculated as the sum over the “wiped area”, i.e. the sum of all nodes which lie to the left of the windscreen wiper.

Establishing the relationship between wipers and generating chains of a Souslin algebra is quite straight forward. Part b) of the next lemma, which is a refinement of Lemma 2.3 requires a second catch-up argument in order to achieve the denseness property (ii) of the definition of wipers.

Lemma 2.5 (cf. [SF08, 2.2.4]). *Let \mathbb{B} be a Souslin Algebra with Souslinisation T .*

- a) *If W is a wiper on T , then K_W is a generating chain of \mathbb{B} .*
- b) *Let K be a generating chain of \mathbb{B} . Then there is a club $C \subseteq \omega_1$, s.t. there is a wiper W on $T \restriction C$, inducing K , i.e. with $K = K_W$ as above.*

By this correspondence we also see that a small Souslin algebra always has exactly 2^{\aleph_1} maximal chains.

3. COMPLETE SUBALGEBRAS AND TREE EQUIVALENCE RELATIONS

In order to represent complete, atomless subalgebras of a Souslin algebra we will use certain equivalence relations defined on a Souslinisation. The basic ideas for this method of representation can already be found in work of Jech and Jensen from around 1970. Here we will use the notions developed in [SF08]² and [SF09] which are more appropriate for our purpose, that of rendering all Souslin subalgebras isomorphic to the superalgebra. In particular, we will take a topological point of view towards countable trees which simplifies matters significantly. (Cf. [SF09, §§2,5] for a more detailed account on this.)

²Note that t.e.r.s as defined here and in [SF09] are called *decent* t.e.r.s in [SF08]. The property of decency of [SF08, Def.1.1.3] refers to what is called honesty in the present text and in [SF09].

Definition 3.1. Let T be a normal, \aleph_0 -branching tree.

- a) We say that an equivalence relation \equiv on T is a *tree equivalence relation* if \equiv respects levels (i.e., \equiv refines $T \otimes T$), is compatible with the tree order of T (i.e., $\text{ht } s = \text{ht } r$ and $s < t \equiv u > r$ imply $s \equiv r$) and honest (i.e. for each triple $s, s', t \in T$ such that $s < s'$ and $s \equiv t$ we either find a node $t' > t$ such that $s' \equiv t'$ or there is no immediate successor of t equivalent to $s' \restriction (\text{ht}(s) + 1)$, the immediate successor of s below s') and the quotient T/\equiv is a normal and \aleph_0 -branching tree with the induced order.
- b) A t.e.r. \equiv on T is said to be
- *nice* if for all triples $s, s', t \in T$ with $s \equiv t$ and $s < s'$ there is some $t' > t$ equivalent to s' ;
 - ∞ -*nice* if it is nice and for node s and every \equiv -class t/\equiv the intersection $T(s) \cap (t/\equiv)$ is either empty or infinite. (Equivalently we could require only $\text{succ}(s) \cap (t/\equiv)$ to be infinite.)

A t.e.r. \equiv on a Souslinisation T of \mathbb{B} represents the atomless and complete subalgebra $\mathbb{A}_\equiv = \langle \sum t/\equiv \mid t \in T \rangle^{\text{cm}}$ of $\mathbb{B} = \text{RO } T$. (We denote the \equiv -class of $t \in T$ by t/\equiv .) A subalgebra of \mathbb{B} is called (∞) -*nice*, if it is represented by a (∞) -nice t.e.r. on some Souslinisation of \mathbb{B} . Since niceness descends to the restriction to a club set of levels, this is independent from the choice of the Souslinisation. Nice subalgebras are much easier to handle than general subalgebras. But as chain homogeneous Souslin algebras are necessarily homogeneous (cf. Proposition 8.1) and every homogeneous, small Souslin algebra also has non-nice, atomless and complete subalgebras ([SF09, Thms.3 & 4]) we also have to consider this general case in our constructions.

Lemma 3 of [SF09] states that for every Souslin algebra \mathbb{B} with a Souslinisation T and every complete and atomless subalgebra \mathbb{A} of \mathbb{B} there is a t.e.r. \equiv that represents \mathbb{A} on $T \restriction C$ for some club C of ω_1 . If \mathbb{A} is nice or ∞ -nice then \equiv can be chosen in a corresponding fashion.

A maximal branch b of a tree T is called *cofinal* if its order type with respect to $<_T$ coincides with the height of T . We let

$$[T] = \{b \mid b \text{ is a cofinal branch of } T\}.$$

For $s \in T$ set $\hat{s} := \{b \in [T] \mid s \in b\}$. We consider $[T]$ as a topological space with the base $\{\hat{s} \mid s \in T\}$. Then for a normal tree T of countable limit height α with only countable levels the space $[T]$ is a perfect Polish space, i.e., a separable and completely metrizable space without isolated points.

It is clear that every tree homomorphism $\varphi : T \rightarrow S$ induces a continuous mapping $\bar{\varphi} : [T] \rightarrow [S]$ and that for every club $c \subseteq \text{ht } T$ we have a natural homeomorphism between $[T]$ and $[T \restriction c]$ since every cofinal branch of T is uniquely determined by its

intersection with $T|c$. If T carries a t.e.r. \equiv , then an equivalence relation on $[T]$ is induced which we also will denote by \equiv .

We now come to state the key property and the associated lemma we use for constructions of Souslin algebras with tightly controlled subalgebras.

Definition 3.2. Let X be a topological space and \equiv an equivalence relation on X . We say that a subset $N \subseteq X$ is *suitable for \equiv* if for every element $x \in X$ the intersection of its \equiv -class x/\equiv with N is either empty or dense in x/\equiv (with the subspace topology).

In order to add a new limit level to a countable tree T we will choose a countable, dense subset Q of $[T]$ and assign to every member $x \in Q$ a bounding limit node. A given a t.e.r. \equiv will be extended to the new level if and only if the set Q is suitable for \equiv . Other wise it would fail to be honest. We will always succeed in extending \equiv if we are allowed to choose the members of Q from a comeagre subset M of $[T]$ that is suitable for \equiv . The Reduction Lemma states that we will find such a set M .

Lemma 3.3 (Reduction Lemma, [SF09, Lemma 7]). *Let T be a countable, normal and \aleph_0 -branching tree of limit height carrying a t.e.r. \equiv , and let $M \subseteq [T]$ be comeagre. Then there is a comeagre subset $M' \subseteq M$ which is suitable for \equiv .*

For a monotone map $\varphi : S \rightarrow T$ between trees we denote the mapping between the branch spaces $[S] \rightarrow [T]$, $x \mapsto \varphi''x$ by $\overline{\varphi}$. We will say that φ induces $\overline{\varphi}$. Extending a tree mapping φ to a new limit level requires, in the notation used above the Reduction Lemma, that the set Q is closed under the application of $\overline{\varphi}$. The two following results establish that the Reduction Lemma can also be applied to achieve this situation.

Lemma 3.4. *For a normal, countable tree T of limit height equipped with a t.e.r. \equiv consider the canonical tree epimorphism*

$$\pi : T \rightarrow T/\equiv, \quad \pi(t) = t/\equiv.$$

Let the set $Y := \{y \in [T/\equiv] \mid (\exists x \in [T]) y = \pi''x\}$ be equipped with the subspace topology inherited from $[T/\equiv]$. Then the induced map

$$\overline{\pi} : [T] \rightarrow Y, \quad x \mapsto \pi''x$$

is an open and surjective mapping and Y is a comeagre subset of $[T]$.

Proof. Surjectivity of $\overline{\pi}$ as in the statement of the lemma is trivial. In order to prove that $\overline{\pi}$ is an open mapping, we show that the images of the basic open sets \hat{t} , $t \in T$ are open in Y . So let t be a node of T . We prove the equation $\overline{\pi}''\hat{t} = \bigcup \{\widehat{\pi(r)} \mid r \in \text{succ}(t)\} \cap Y$. Letting $\overline{\pi}(x) = \pi''x$ with $t \in x \in [T]$ be a member of the left hand side we choose r to be the unique immediate successor of t in x and see that this branch is also a member of the right hand side.

For the converse inclusion let $y \in Y$ be in $\widehat{\pi(r)}$ for some immediate successor r of t . We have to find a branch $x \in [T]$ through t with $\pi(x) = y$. As y is in Y it has a π -preimage $x^* \in [T]$. Letting $\gamma := \text{ht}(t)$, take s to be the node of x^* on level γ and s' its immediate successor in x^* . Then clearly $s \equiv t$ and $s' \equiv r$. Because of \equiv 's honesty, there are on every level $\delta > \gamma$ successors of t which stand in relation \equiv to a member of x^* . The set of all these nodes forms (together with t and its predecessors) a subtree T^* of T which has a cofinal branch x by König's Lemma, because in any case we can reduce T^* to a club set of levels of order type ω in which we find a finitely branching subtree of the same height. Clearly we have $\pi(x) = \pi(x^*) = y$, as desired.

Finally, Y is separable, metrizable and the image of the Polish space $[T]$ under the open mapping π , so Y is Polish by a theorem of Sierpiński (cf. [Kec95, 8.19]) and thus comeagre in $[S]$ by Choquet's Theorem (cf. [Kec95, 8.17.ii]) being a dense subset of $[S]$ that contains a Polish space as a subset. \square

Proposition 3.5. *Let X and Y be Polish spaces and $h : X \rightarrow Y$ a continuous mapping, such that $h''X$ is comeagre in Y and the right-hand side restriction of h to its image, i.e. $h : X \rightarrow h''X$ is an open mapping.*

- a) *If $D \subset Y$ is meagre, then $h^{-1''}D$ is meagre as well.*
- b) *If $M \subset X$ is comeagre, then $h''M$ is comeagre as well.*

Proof. The proof of part a) is straight forward. For part b) we can assume without loss of generality, that M is a dense G_δ -subset of X . Then clearly $h''M$ is an analytic subset of Y . So, by a theorem of Lusin and Sierpinski (cf. [Jec03, Thm.11.18.b] or [Kec95, Thm.21.6]), $h''M$ has the Baire property, i.e. there is an open subset U of Y , such that the symmetric difference of $h''X$ and U is meagre. In particular the sets $h''X \setminus U$ and $U \setminus h''X$ are meagre. If we can show, that U is dense in Y , then the proof is finished.

So assume to the contrary that there is an open subset V of Y which is disjoint from U . Then $V \cup h''X$ is meagre and so is $M \cap h^{-1''}V = h^{-1''}(V \cap h''M)$ by part a). But since M is comeagre and $h^{-1''}V$ is open, their intersection cannot be meagre — contradiction! \square

We combine the Reduction Lemma and the last two propositions in order to get the result which is appropriate for our use in the constructions.

Corollary 3.6. *Let T be a countable, normal and \aleph_0 -branching tree of limit height α . Let H be a countable set of triples $h = (c_h, \equiv_h, \varphi_h)$ where c_h is a club subset of α , \equiv_h is a t.e.r. on $T \restriction c_h$ (with associated canonical mapping π_h) and $\varphi_h : (T \restriction c_h) / \equiv_h \rightarrow T \restriction c_h$ is an isomorphism. Let furthermore I be a countable set of pairs $i = (c_i, \equiv_i)$ such that c_i is club in α and \equiv_i is an ∞ -nice t.e.r. on $T \restriction c_i$. If M is a comeagre subset of $[T]$ then there is a comeagre subset N of M , such that*

- (1) for all $h \in H$, N is suitable for \equiv_h and $(\overline{\varphi}_h \circ \overline{\pi}_h)''N = N$;
- (2) N is suitable for all \equiv_i where $i \in I$.

Proof idea. As images and pre-images of comeagre sets under the mappings from H are all comeagre, repeated intersections deliver N . \square

Our last technical lemma on t.e.r.s concerns nested ∞ -nice t.e.r.s and has a straight forward proof. If \equiv_1 refines \equiv_0 as an equivalence relation on some set M , we define the equivalence relation $\equiv_{0/1}$ on M/\equiv_1 by

$$(x/\equiv_1) \equiv_{0/1} (y/\equiv_1) : \iff x \equiv_0 y$$

for $x, y \in M$.

Lemma 3.7. *Assume we are given two ∞ -nice t.e.r.'s \equiv_0 and \equiv_1 on T which is normal, \aleph_0 -branching and of countable limit height, such that \equiv_0 is refined by \equiv_1 and the t.e.r. $\equiv_{0/1}$ on the quotient tree T/\equiv_1 induced by \equiv_0 is ∞ -nice. Let furthermore M be a comeagre subset of $[T]$ that is suitable for both \equiv_0 and \equiv_1 . Then M/\equiv_1 is suitable for $\equiv_{0/1}$.*

4. VARIATIONS OF KUREPA'S LEMMA

A striking property of normal trees is formulated in Kurepa's Isomorphism Lemma, cf. [Kur35, p.102]. The result and its proof are well-known, but since we will use some variations of the argument later on, we also sketch the proof.

Lemma 4.1. *Let S, T be two normal κ -branching trees, $\kappa \leq \omega$, of the same limit height $\alpha < \omega_1$ with countable levels only. Then $S \cong T$.*

Proof. So let the height of S and T be a countable limit ordinal α . Choose countable and dense sets $X \subset [S]$ and $Y \subset [T]$ and enumerate them by $(x_i \mid i \in \omega)$ and $(y_i \mid i \in \omega)$ respectively.

We give a back-and-forth-construction of a bijective mapping $f : X \rightarrow Y$ which lifts to a tree isomorphism $\varphi : S \rightarrow T$. Define $f(x_0) = y_0$ and $\varphi(x_0 \upharpoonright \gamma) = y_0 \upharpoonright \gamma$ for all $\gamma < \alpha$.

As the forth-argument is completely analogous, we only explain how the back-step works. Let i be minimal such that $f^{-1}(y_i)$ has not yet been fixed, and pick the minimal γ such that $\varphi^{-1}(y_i \upharpoonright \gamma)$ has not yet been defined.

This γ is a successor ordinal, say $\gamma = \delta + 1$, because S and T are assumed to be normal trees.

Now choose an immediate successor s of $\varphi^{-1}(y_i \upharpoonright \delta)$ such that $\varphi(s)$ has not yet been defined. Such a node s exists by the choice of γ . Finally let j be minimal such that $s \in x_j$. Then $f(x_j)$ has not yet been defined, but we set $f(x_j) = y_i$ and $\varphi(x_j \upharpoonright \gamma) = y_i \upharpoonright \gamma$. This is consistent with the choices met for f and φ so far.

After every step of the construction φ is a partial isomorphism between S and T . By the choice of X and Y , the union of these partial isomorphisms is bijective, so in the end $\varphi : S \rightarrow T$ is an isomorphism. \square

The argument also shows that for any $\gamma < \alpha$ and every isomorphism φ_0 between $S \restriction (\gamma + 1)$ and $T \restriction (\gamma + 1)$ there is an extension $\varphi_0 \subset \varphi : S \cong T$.

The denseness requirement in Definition 2.4 of *wipers* enables us to combine the argument from the proof of Kurepa's Isomorphism Lemma with that of Cantor's for the \aleph_0 -categoricity of the dense linear orders in order to construct isomorphisms between wipers.

Lemma 4.2. *Let T and S be two countable, \aleph_0 -branching, and normal trees of the same height $\alpha < \omega_1$ and let $W_0 = \langle \prec_\gamma \mid \gamma < \alpha \rangle$ be a wiper on T and $W_1 = \langle <_\gamma \mid \gamma < \alpha \rangle$ a wiper on S . Furthermore let $\beta < \alpha$ and φ' be an isomorphism from $T \restriction (\beta + 1)$ onto $S \restriction (\beta + 1)$, such that for all $\gamma \leq \beta$ and $s, t \in T_\gamma$ we have*

$$s \prec_\gamma t \Leftrightarrow \varphi'(s) <_\gamma \varphi'(t).$$

Then there is an isomorphism φ between T and S extending φ' , such that for all $\gamma < \alpha$ and $s, t \in T_\gamma$ we have

$$s \prec_\gamma t \Leftrightarrow \varphi'(s) <_\gamma \varphi'(t).$$

Proof. We refer to the proof of Kurepa's Isomorphism Lemma 4.1 and describe the only manipulation: When it comes to choosing of $\varphi^{-1}(s)$, this choice has to respect the wipers, which is always possible by the denseness requirement (ii) of Definition 2.4.a). \square

Our last Kurepa-style lemma states that isomorphisms between quotient trees by ∞ -nice t.e.r.s can be lifted to isomorphisms of the underlying trees. In order to prove it, we first formulate a regularity property of subtrees induced by an ∞ -nice t.e.r.

Proposition 4.3. *Let T be a normal, \aleph_0 -branching tree of countable limit height carrying an ∞ -nice t.e.r. \equiv . Then for every cofinal branch b of T the set $\{s \in T \mid (\exists r \in b) s \equiv r\}$ is an \aleph_0 -branching, normal tree.*

Proof. Unique limits are inherited from T , infinite branching follows from the ∞ -condition, successors in every higher level follow from niceness. \square

Proposition 4.4. *Let T be a normal, \aleph_0 -branching tree of limit height $\alpha < \omega_1$ and let \equiv and \sim be ∞ -nice t.e.r.'s on T . Let γ be equal to α or else be a successor ordinal below α . Let φ' be an isomorphism between $(T \restriction \gamma)/\equiv$ and $(T \restriction \gamma)/\sim$. Then there is an automorphism φ of T that carries \equiv to \sim , s.t. the induced map on T/\equiv is an isomorphism onto T/\sim that extends φ' .*

Proof. First use Lemma 4.1 to extend φ' to an isomorphism between T/\equiv and T/\sim in the case that $\gamma < \alpha$. So we can assume $\alpha = \gamma$.

We now give a back-and-forth-argument which lifts the isomorphism φ' to an automorphism φ of T . Enumerate T in order type ω by s_0, s_1, \dots . For the induction step in the forth-direction let i be the minimal index i for which $\varphi(s_i)$ has not yet been defined. Our choice for $\varphi(s_i)$ has to respect the tree order and the equivalence relations. So let s be the predecessor of maximal height whose image under φ is already determined by the choices met earlier in the construction, where $s = s_i$ is allowed. In the case that $s <_T s_i$, we want to pick a node $t \in \varphi'(s_i/\equiv)$ above $\varphi(s)$ which exists by the niceness of \sim . We furthermore require that t has not yet been assigned as some $\varphi(s_j)$. This choice is possible due to the ∞ -part in the ∞ -niceness of \sim .

For the back-step we replace in the above argument all φ and φ' by φ^{-1} and φ'^{-1} . \square

5. STATEMENT OF THE MAIN RESULTS

Definition 5.1. Let \mathbb{B} be a Souslin algebra. By *continuous, ∞ -nice chain of subalgebras of \mathbb{B}* we denote a sequence $(A_\nu \mid \nu < \beta)$ of ∞ -nice subalgebras of \mathbb{B} such that

- (i) A_μ is an ∞ -nice-subalgebra of A_ν for $\mu < \nu < \beta$,
- (ii) for limit $\alpha < \beta$ the algebra A_α is completely generated by the preceding members A_ν , $\nu < \alpha$ of the chain, i.e.

$$A_\alpha = \left\langle \bigcup_{\nu < \alpha} A_\nu \right\rangle^{\text{cm}}.$$

Theorem 5.2. Assume \diamond^+ . Then there is a small, chain homogeneous Souslin algebra \mathbb{B} . Furthermore, the construction can be refined to achieve the following additional homogeneity properties.

- a) For every isomorphism $\varphi' : A \cong A'$ of ∞ -nice subalgebras of \mathbb{B} there is an automorphism φ of \mathbb{B} extending φ' .
- b) There is a continuous, ∞ -nice chain $(A_i \mid i < \omega_1)$ of subalgebras of \mathbb{B} such that:
 - (i) $\bigcup_{j < \omega_1} A_j = \mathbb{B}$;
 - (ii) for every continuous, ∞ -nice chain $(C_\nu \mid \nu < \lambda + 1)$ of \mathbb{B} (λ a countable limit) there is an isomorphism $\varphi : A_\lambda \rightarrow C_\lambda$ such for all $\nu < \lambda$ the restriction of φ to A_ν is an isomorphism onto C_ν .

The additional homogeneity properties stated in a) and b) of Theorem 5.2 have as primary goal their use in the construction which proves the following theorem. It answers a question of Stevo Todorćević, who asked on the occasion of a

talk the author gave on chain homogeneous Souslin algebras at Toposym X in 2006 in Prague, whether there are also big, chain homogeneous Souslin algebras.

Theorem 5.3. *Assume \diamond^+ and let \mathbb{B} be a chain homogeneous Souslin algebra as in Theorem 5.2. Then there is a big, chain homogeneous Souslin algebra \mathbb{B}^* all of whose subalgebras which are small Souslin algebras are isomorphic to \mathbb{B} .*

As the proof of this latter theorem is rather short, we give it right away.

Proof of Theorem 5.3 from Theorem 5.2. We realise the big and chain homogeneous Souslin algebra \mathbb{B}^* as the union of an increasing chain of small, chain homogeneous Souslin algebras $(\mathbb{B}_\alpha \mid \alpha < \omega_2)$. All the small algebras \mathbb{B}_α on this chain are isomorphic to the initial algebra $\mathbb{B}_0 := \mathbb{B}$, which we take from Theorem 5.2. The properties of \mathbb{B}^* as stated in the theorem then follow from general principles. As all the \mathbb{B}_α satisfy the c.c.c. and \mathbb{B}^* is defined as their direct limit, \mathbb{B}^* also satisfies the c.c.c. and is therefore complete. Distributivity and chain homogeneity are also inherited.

As above, denote by $(\mathbb{A}_i \mid i < \omega_1)$ the continuously increasing chain of ∞ -nice subalgebras of \mathbb{B} . For the successor step, if \mathbb{B}_α is given, choose any isomorphism $\varphi_\alpha : \mathbb{B}_\alpha \cong \mathbb{A}_0$. Next choose a Souslin algebra $\mathbb{B}_{\alpha+1}$ extending \mathbb{B}_α , such that there is an extension $\varphi_{\alpha+1}$ of φ_α and witnessing $\mathbb{B}_{\alpha+1} \cong \mathbb{B}$. This extension exists by part (a) of Theorem 5.2.

If $\lambda < \omega_2$ is of countable cofinality, we choose a normal sequence $(i_\nu \mid \nu < \mu)$ with $\sup_{\nu < \mu} i_\nu = \lambda$ for some countable limit ordinal μ . Inductively construct an increasing chain $(\psi_\nu \mid \nu < \mu)$ of isomorphisms $\psi_\nu : \mathbb{A}_\nu \cong \mathbb{B}_{i_\nu}$. Then choose \mathbb{B}_λ as a super-algebra of

$$\bigcup_{\nu < \mu} \mathbb{B}_{i_\nu} = \bigcup_{\alpha < \lambda} \mathbb{B}_\alpha$$

isomorphic to \mathbb{A}_μ by some extension ψ_μ of $\bigcup_{\nu < \mu} \psi_\nu$.

If λ has uncountable cofinality we simply let $\mathbb{B}_\lambda = \bigcup_{\alpha < \lambda} \mathbb{B}_\alpha$, which in this case is the direct limit. Then we see that $\mathbb{B}_\lambda \cong \mathbb{B}_0$ for $\lambda < \omega_2$ by choosing a cofinal sequence $(i_\nu \mid \nu < \omega_1)$ and recursively choosing a chain of isomorphisms $\psi_\nu : \mathbb{A}_\nu \cong \mathbb{B}_{i_\nu}$. This goes through limit stages $\nu < \omega_1$ by property (b.ii) of Theorem 5.2. Iterating up to $\alpha = \omega_2$ finishes the construction of \mathbb{B}^* . \square

6. FINAL PREPARATIONS

In our Souslin tree construction we want to build additional objects (mappings, tree isomorphisms) on club sets of levels of the tree T to be constructed, that relate given objects (e.g. pairs of wipers or t.e.r.s on $T \restriction C$) to each other. During a relevant construction step, initial segments of some of the given objects are proposed by a \diamond^+ -sequence and in order to extend an additional object we need some pointer to

indicate the ordinal stage up to which the recursive construction of the additional object has reached so far. This is the function of ε of the following definition.

Definition 6.1 (the \diamond^+ -machinery). Fix a \diamond^+ -sequence $(S_\alpha)_{\alpha < \omega_1}$.

- a) Let $C_0 := \{\alpha < \omega_1 \mid \omega\alpha = \alpha\}$ be the set of countable fixed points of the left-multiplication with ω .
- b) Let $\alpha \in C_0$. For $x \in \mathcal{P}(\alpha)$ set

$$c(x) = \{\gamma \in C_0 \cap \alpha \mid x \cap \omega(\gamma + 1) \setminus \omega\gamma \neq \emptyset\}.$$

(This will be the set of levels of the tree, to which x refers.)

- c) The *set of relevant guesses for stage α* , G_α is the set of pairs $(x, d) \in S_\alpha \times S_\alpha$ such that $c(x)$, d and $c(x) \cap d$ are club in α and for $\gamma \in d$ the sets $x \cap \gamma$ and $d \cap \gamma$ are in S_γ .
- d) For $(x, d) \in G_\alpha$ let

$$e_{x,d} := \{\gamma \in c(x) \cap d \mid \bigcup (\gamma \cap c(x) \cap d) = \gamma\}$$

be the Cantor-Bendixson derivative of $c(x) \cap d$, i.e., the set of its limit points.

- e) Let $\varepsilon_{x,d} := \bigcup e_{x,d}$. (Note that $\varepsilon_{x,d} = 0$ if $\text{ot}(c(x) \cap d) = \omega$.)

Now if for example $(x, d) \in G_\alpha$ and x codes a pair of wipers on $T \restriction c(x)$, we know that up to stage $\varepsilon_{x,d}$ our recursive construction of the additional object — here: an isomorphism between the wipers given by x — has been invoked and therefore up to this stage this isomorphism has yet been constructed.

Recall Lemma 4.4, which states that an isomorphism between two quotient trees T/\equiv_0 and T/\equiv_1 can be lifted to an automorphism of T if the t.e.r.s are ∞ -nice. In order to get hold of isomorphisms between quotient trees we introduce the notion of an engaging relation. Consider two ∞ -nice t.e.r.'s \equiv_0 and \equiv_1 on $T \restriction c$, with $c \subset \alpha$ club, and an isomorphism $\varphi : (T \restriction c)/\equiv_0 \rightarrow (T \restriction c)/\equiv_1$ between the quotient trees. Then φ naturally induces a relation Φ on $T \restriction c$, that consists of the pairs $s, t \in T \restriction c$ with $\varphi(s/\equiv_0) = t/\equiv_1$. The properties of such a relation Φ are captured in the following definition.

Definition 6.2. We say that a relation Φ on a tree T satisfying points 1-4) below is *engaging*.

- (1) There is a set c_Φ such that for all $s, t \in T$ we have $s\Phi t$ only if $\text{ht}(s) = \text{ht}(t) \in c_\Phi$,
- (2) the left-induced relation $\Phi^0 := \{(s, s') \in (T \restriction c)^2 \mid (\exists t)s\Phi t \text{ and } s'\Phi t\}$ is an ∞ -nice t.e.r. on $T \restriction c_\Phi$,
- (3) the right-induced relation $\Phi^1 := \{(t, t') \in (T \restriction c)^2 \mid (\exists s)s\Phi t \text{ and } s\Phi t'\}$ also is an ∞ -nice t.e.r. on c_Φ

- (4) Φ induces an isomorphism φ_Φ between $(T \restriction c)/\Phi^0$ and $(T \restriction c)/\Phi^1$ via $\varphi_\Phi(s/\Phi^0) = t/\Phi^1$ for any t with $s\Phi t$.

It is clear that the relation Φ considered above the last definition is indeed engaging with $c_\Phi = c$, and Φ^0 is \equiv_0 while Φ^1 is \equiv_1 .

Finally we define a certain sequence of club sets of ω_1 which will act as the family of supporting sets of levels for the t.e.r.s that represent the increasing sequence $(\mathbb{A}_i \mid i < \omega_1)$ if ∞ -nice subalgebras. Recall the definition of C_0 , the set of infinite fixed points of the left hand ordinal multiplication with ω in ω_1 :

$$C_0 := \{\alpha < \omega_1 \mid \alpha \neq 0, \omega\alpha = \alpha\}.$$

Inductively define C_{i+1} for $i < \omega_1$ to be the Cantor-Bendixson-derivative of C_i , and for limit ordinals i let C_i be the intersection of the C_j defined so far:

$$C_{i+1} := \{\alpha \in C_i \mid \sup(C_i \cap \alpha) = \alpha\} \quad \text{and} \quad C_i = \bigcap_{j < i} C_j \text{ for limit } i.$$

We list some properties of the sequence $(C_i)_{i < \omega_1}$ used in the construction below.

- (a) all the C_i are club in ω_1 ,
- (b) the sequence is continuously decreasing and has an empty intersection, hence there is for every $\alpha \in C_0$ a unique $i = i(\alpha)$ with $\alpha \in C_i \setminus C_{i+1}$,
- (c) every $\alpha \in C_0$ has a direct predecessor in $C_{i(\alpha)}$, call it α^- ,
- (d) for limit $i < \omega_1$, the minimum of C_i is the supremum of the minima of the C_j for $j < i$.

7. PROOF OF THEOREM 5.2

Before we step into the details of the construction, we give a brief sketch of the main steps to be carried out at one stage of the recursion. We use the principle \diamond^+ during the construction in order to get hold of all kinds of objects we are interested in, which are

- pairs of wipers that represent pairs of generating chains;
- t.e.r.s that represent Souslin subalgebras;
- engaging relations that represent isomorphism between ∞ -nice subalgebras.

All of these are relations on $T \restriction C$ for some club C of ω_1 . In a relevant stage of the construction, say stage $\alpha \in C_0$, we have the tree $T \restriction \alpha$ at hand and carry out five steps:

- (1) We collect all pairs $(x, d) \in G_\alpha$ that correspond to any of the above items. If necessary, we extend the mappings associated to these objects which were chosen in earlier stages of the construction by virtue of Kurepa's Lemma and its variations up to below T_α .

- (2) We also collect the relevant t.e.r.s which represent the members of our increasing sequence of ∞ -nice subalgebras.
- (3) Now we consider the induced relations and mappings on the Polish space $[T \restriction \alpha]$ and find with the aid of the Reduction Lemma a comeagre subset $N \subset [T \restriction \alpha]$ that is suitable for all relevant equivalence relations and is equal to its image under all relevant mappings: $\bar{p}''N = N$.
- (4) Finally we choose a countable, dense subset Q of N with the same properties corresponding to the objects of interest: suitable for the equivalence relations and $\bar{p}''Q = Q$. The members of Q represent the nodes in our new limit level T_α of the tree T .
- (5) If necessary we extend the t.e.r. which represents to relevant member of the ∞ -nice chain.

7.1. Setting the stage, casting and styling. We now begin and inductively, i.e., level by level, construct a tree-order $<_T$ on the set ω_1 , such that the resulting tree T will be a normal and \aleph_0 -branching Souslin tree. The Souslin algebra to be constructed will be the regular open algebra of $T = (\omega_1, <_T)$.

Let 0 be the root of T and in every successor step fix \aleph_0 distinct direct successors for each maximal node in such a way that for every $\alpha < \omega_1 \setminus \{0\}$ the level T_α consists of the next ω many ordinals not yet used in the construction. So we have $T_1 = \omega \setminus \{0\}$, $T_n = \omega n \setminus \omega(n-1)$ for all natural numbers $n \geq 2$ and finally $T \restriction \alpha = \omega \alpha$ for all infinite, countable ordinals α .

We fix a \diamond -sequence $(R_\nu)_{\nu < \omega_1}$, a \diamond^+ -sequence $(S_\nu)_{\nu < \omega_1}$ and a bijection $g : \omega_1 \rightarrow (2 \times \omega_1 \times \omega_1)$ with $g''\lambda = 2 \times \lambda \times \lambda$ for all limit ordinals λ . Let for $i \in 2$ be g_i the concatenation of g and the projection onto the fibre over i . We will use g for coding t.e.r.s and pairs of wipers as sets of ordinals.

Now let $\alpha < \omega_1$ be a limit ordinal. By the inductive assumption we have so far constructed a normal tree order $(T \restriction \alpha, <_T)$ on the supporting set $\omega \alpha$.

If $\alpha < \omega \alpha$ we simply choose a countable dense subset Q_α of $[T \restriction \alpha]$ and embed Q_α onto $\omega(\alpha + 1) \setminus \omega \alpha$, i.e., we choose a bijection between Q_α and $\omega(\alpha + 1) \setminus \omega \alpha$ and extend $<_T$ on $\omega(\alpha + 1)$ in the obvious way.

If $\alpha = \omega \alpha$, we want to choose a countable dense subset Q_α of $[T \restriction \alpha]$, too, but this time our set also has to seal certain maximal antichains of $T \restriction \alpha$ and extend enough tree isomorphisms. Recall the definitions from the last section concerning the use of the \diamond^+ -sequence $(G_\alpha, c(x), \varepsilon_{x,d}, \text{etc.})$.

From now on, if $(x, d) \in G_\alpha$ is fixed, we write c for $c(x)$. Denote by E_α the subset of $S_\alpha \times S_\alpha$ that consists of all pairs $(x, d) \in G_\alpha$ such that $g_0''x$ and $g_1''x$ are wipers $W_0 = \langle \prec_\gamma : \gamma \in c(x) \rangle$ and $W_1 = \langle \prec_\gamma : \gamma \in c(x) \rangle$ on $T \restriction c(x)$ respectively. So $(x, d) \in E_\alpha$ if x codes a pair of wipers on $T \restriction c$ and is guessed correctly by the \diamond^+ -sequence along with a club d on the members of d itself.

Now we take care of the homogeneity property stated above as (a). We define

$$E'_\alpha := \{(x, d) \in G_\alpha \mid x \text{ codes an engaging relation } \Phi_x \text{ on } T \restriction c(x)\}.$$

Every $(x, d) \in E'_\alpha$ induces a pair of ∞ -nice t.e.r.s \equiv_0^x and \equiv_1^x and an isomorphism φ'_x between $(T \restriction c)/\equiv_0^x$ and $(T \restriction c)/\equiv_1^x$.

The guesses of the \diamond^+ -sequence for t.e.r.s are collected in the set F_α :

$$F_\alpha := \{(x, d) \in G_\alpha \mid x \text{ codes a t.e.r. on } c(x)\}.$$

Here we code e.g. with respect to the map g_0 . For $(x, d) \in F_\alpha$ let \equiv_x denote the t.e.r. which is coded by x . It induces the \aleph_0 -branching, normal tree $(T \restriction c)/\equiv_x$.

Keep in mind that $c \cap d$ is unbounded in α if $(x, d) \in E_\alpha$ or F_α by the definition of the relevant stages set G_α .

The inductive hypothesis is that $T \restriction \alpha$ is a countable and \aleph_0 -branching tree of height α and

- (1) for every pair $(x, d) \in E_\alpha$ there is a \subset -chain $\langle \varphi_{x \cap \gamma, d \cap \gamma} : \gamma \in e_{x,d} \rangle$ and each $\varphi_{x \cap \gamma, d \cap \gamma}$ is a tree automorphism of $T \restriction (c \cap d \cap \gamma + 1)$ that was fixed in induction step $\gamma \in e_{x,d}$ and which carries $W_{x,d}^0 \restriction c \cap d \cap \gamma + 1 = \langle \prec_\delta : \delta \in c \cap d \cap \gamma + 1 \rangle$ to $W_{x,d}^1 \restriction c \cap d \cap \gamma + 1 = \langle <_\delta : \delta \in c \cap d \cap \gamma + 1 \rangle$. In short: $\varphi_{x \cap \gamma, d \cap \gamma}$ is an isomorphism of wipers between $W_{x,d}^0 \restriction c \cap d \cap \gamma + 1$ and $W_{x,d}^1 \restriction c \cap d \cap \gamma + 1$ for all $\gamma \in e_{x,d}$;
- (2) for every pair $(x, d) \in E'_\alpha$ there is an automorphism of $T \restriction c \cap d \cap \varepsilon_{x,d}$ which extends φ'_x ;
- (3) for every pair $(x, d) \in F_\alpha$ there is a \subset -chain $\langle \psi_{x \cap \gamma, d \cap \gamma} : \gamma \in e_{x,d} \rangle$ of tree isomorphisms $\psi_{x \cap \gamma, d \cap \gamma} : T \restriction (c \cap d \cap \gamma + 1)/\equiv_x \rightarrow T \restriction (c \cap d \cap \gamma + 1)$ each of them fixed in induction step $\gamma \in e_{x,d}$.

(Recall that $c = c(x)$ and $e_{x,d}$ is the set of limit points of $c \cap d$.) For those (x, d) in E_α, E'_α or in F_α but with $\varepsilon_{x,d} < \alpha$ we need to choose extensions for the maps granted by the inductive hypothesis as follows.

Fix $(x, d) \in E_\alpha$. If $\varepsilon_{x,d} = \alpha$ let

$$\varphi_{x,d} := \bigcup_{\gamma \in e_{x,d}} \varphi_{x \cap \gamma, d \cap \gamma}.$$

Otherwise extend $\varphi_{x \cap \varepsilon_{x,d}, d \cap \varepsilon_{x,d}}$ by Proposition 4.2 to some isomorphism $\varphi_{x,d}$ between $W_{x,d}^0 \restriction c \cap d$ and $W_{x,d}^1 \restriction c \cap d$.

For $(x, d) \in E'_\alpha$ use Lemma 4.4 to lift φ_{Φ_x} to an automorphism $\varphi_{x,d}$ of $T \restriction c(x) \cap d$ carrying Φ^0 to Φ^1 which extends φ'_x .

Let $(x, d) \in F_\alpha$ and set $\psi_{x,d} := \bigcup_\gamma \psi_{x \cap \gamma, d \cap \gamma}$ if $\varepsilon_{x,d} = \alpha$. Otherwise extend the union of the chain by Kurepa's Lemma 4.1 to some isomorphism

$$\psi_{x,d} : (T \restriction c \cap d)/\equiv_x \rightarrow T \restriction c \cap d.$$

7.2. Arranging the scene for the ∞ -nice chain. We now describe how to embed the increasing ω_1 -sequence of ∞ -nice subalgebras \mathbb{A}_i in \mathbb{B} . Recall the definition of the decreasing sequence $(C_i \mid i \in \omega_1)$ of club subsets of ω_1 as well as the derived definitions of $i(\alpha)$ and α^- . The ∞ -nice t.e.r. \equiv_i representing \mathbb{A}_i will be defined on $T \restriction C_i \cup \{0\}$ in the course of the construction of T . The main requirements to meet are:

- (1) \equiv_i is ∞ -nice for all $i < \omega_1$,
- (2) for $i > j$ the restriction of \equiv_j to $T \restriction C_i$ is refined by \equiv_i in a way such that the induced t.e.r. $\equiv_{j/i}$ on the normal Souslin tree $(T \restriction C_i) / \equiv_i$ is ∞ -nice,
- (3) for limit $i < \omega_1$ we want to have $\mathbb{A}_i = \langle \bigcup_{j < i} \mathbb{A}_j \rangle^{\text{cm}}$, so \equiv_i shall be the conjunction of the \equiv_j for $j < i$ in this case:

$$s \equiv_i t :\Leftrightarrow (\forall j < i) s \equiv_j t.$$

For any i , on level $T_0 = \{\text{root}\}$ the relation \equiv_i is of course trivial. On level $T_{\min C_i}$ we define \equiv_i to be the identity, i.e., $s \equiv_i t$ if and only if $s = t$ for $s, t \in T_{\min C_i}$. This is a minor violation of the ∞ -niceness requirement we posed on \equiv_i . But this is easily remedied by deleting $\min C_i$ from the club set C_i . On the other hand, by this convention we directly see that in the end $\bigcup \mathbb{A}_i$ will be a dense subset of \mathbb{B} , because $\{\min C_i \mid i < \omega_1\}$ is unbounded in ω_1 .

In level $\alpha \in C_0$ we have that for all $j < i := i(\alpha)$ the set $C_j \cap \alpha$ is club in α and the t.e.r. \equiv_j on $T \restriction (C_j \cap \alpha)$ has by normality of the quotient tree a unique t.e.r.-extension to T_α . So, to satisfy the niceness condition for the t.e.r.s \equiv_j with $j < i(\alpha)$, level T_α has to be chosen suitably with the aid of the Reduction Lemma 3.3.

Let now $\alpha \in C_0$ be such that $i = i(\alpha)$ is a limit ordinal. For $j < i$ define \equiv_j on $[T \restriction \alpha]$ as above and let for $x, y \in [T \restriction \alpha]$

$$x \equiv_i y : \Leftrightarrow (\forall j < i) x \equiv_j y.$$

This is the coarsest possibility to extend \equiv_i to T_α and the only one by our requirement that $\bigcup_{j < i} \mathbb{A}_j$ completely generate \mathbb{A}_i .

We now show that the relation \equiv_i on $[T \restriction \alpha]$ is induced by a (single) ∞ -nice t.e.r. \simeq . Therefore the Reduction Lemma 3.3 can be applied to \equiv_i . The t.e.r. \simeq will be defined as a diagonal along the \equiv_j for $j < i$.

Fix δ_0 in C_0 between α^- and α , so $j_0 = i(\delta_0) < i$. Then define

$$j_\nu := i(\delta_\nu) \quad \text{and} \quad \delta_{\nu+1} := \min C_{j_{\nu+1}} \setminus \delta_\nu = \min C_{j_\nu+1} \setminus \delta_0$$

and $\delta_\mu := \sup\{\delta_\nu \mid \nu < \mu\}$ for limit ordinals μ with $i(\delta_\nu) < i$ for all $\nu < \mu$. Then the $i(\nu)$ are the ordinals from j_0 up to i . The final δ_μ is just α , and this ordinal $\mu = \text{ot}(i \setminus j_0)$ is a limit.

The set of the δ_ν joined with $\{0\}$ is the club set of α on which we now define the t.e.r. \simeq . For $s, t \in T_{\delta_\nu}$ define

$$s \simeq t : \iff [(\forall \beta < \delta_0)(i(\beta) \leq i \Rightarrow s \restriction \beta \equiv_{i(\beta)} t \restriction \beta) \quad \text{and} \quad s \equiv_{j_\nu} t]$$

Then \simeq is an ∞ -nice t.e.r.: fix $s \simeq t$ on level δ_ξ and consider $s' > s$, where $s \in T_{\delta_\nu}$. In the successor case, letting $\nu = \nu^- + 1$, i.e. $j_\nu = j_{\nu^-} + 1$, this follows from the definition of \equiv_{j_ν} having only classes that are dense subsets of $\equiv_{j_{\nu^-}}$ -classes. To argue for niceness in the limit case we refer to the choice of T_{δ_ν} which assures, that the \equiv_{j_ν} -classes lie densely in the \equiv_{j_ν} -classes while the ∞ -part of ∞ -niceness is trivially satisfied on limit stages when satisfied everywhere below. Since the j_ν are cofinal in i , the ∞ -nice t.e.r. \simeq induces \equiv_i on $[T \restriction \alpha]$.

However, for successor $i(\alpha)$ the definition of $\equiv_{i(\alpha)}$ on T_α involves the choice of the new level T_α . So we continue by giving the rules for the choice of T_α .

7.3. Suitably reducing the area. Now Lemma 3.4 comes into play. Let for $(x, d) \in F_\alpha$

$$\pi_{x,d} : T \restriction c \cap d \rightarrow (T \restriction c \cap d) / \equiv_x, \quad s \mapsto s / \equiv_x$$

be the canonical mapping associated to \equiv_x and define

$$\rho_{x,d} := \psi_{x,d} \circ \pi_{x,d} : T \restriction c \cap d \rightarrow T \restriction c \cap d.$$

Since $\psi_{x,d}$ is an isomorphism, the induced continuous map

$$\bar{\rho}_{x,d} : [T \restriction \alpha] \rightarrow [T \restriction \alpha], \quad b \mapsto \{s \mid (\exists t \in b \cap (T \restriction c \cap d)) s <_T \rho_{x,d}(t)\}$$

has a comeagre image in $[T \restriction \alpha]$ and is an open mapping when the range is restricted to $\bar{\rho}_{x,d}''[T \restriction \alpha]$.

The \diamond -sequence $(R_\nu)_{\nu < \omega_1}$ proposes candidates for maximal antichains in the usual way. If R_α is a maximal antichain of $T \restriction \alpha$ then we have to ensure that each member of T_α is a $<_T$ -successor of some element of R_α . That means Q_α has to be a subset of

$$M_\alpha = \{x \in [T \restriction \alpha] : \exists \gamma < \alpha \ x \restriction \gamma \in R_\alpha\}$$

which is itself an open dense subset of $[T \restriction \alpha]$, because R_α is a maximal antichain. If R_α is not a maximal antichain in $T \restriction \alpha$ we simply set $M_\alpha = [T \restriction \alpha]$. We apply Corollary 3.6 of the Reduction Lemma to the sets $M = M_\alpha$ and

$$H = \{\bar{\varphi}_{x,d} \mid (x, d) \in E_\alpha\} \cup \{\bar{\rho}_{x,d} \mid (x, d) \in F_\alpha\}$$

and

$$I := \begin{cases} \{\equiv_j \mid j < i(\alpha)\}, & \text{if } i(\alpha) \text{ is a successor ordinal} \\ \{\equiv_j \mid j \leq i(\alpha)\}, & \text{if } i(\alpha) \text{ is a limit ordinal} \end{cases}.$$

Since the $E_\alpha \cup E'_\alpha$ -part of H consists of homeomorphisms and the F_α -part is subject to Lemma 3.4, the hypotheses are satisfied. The result is a comeagre subset N_α of M_α , such that

- (1) for all $(x, d) \in E_\alpha \cup E'_\alpha$ we have $\overline{\varphi}_{x,d}'' N_\alpha = N_\alpha$;
- (2) for each $(x, d) \in F_\alpha$, N_α is suitable for \equiv_x and $\overline{\rho}_{x,d}'' N_\alpha = N_\alpha$;
- (3) N_α is suitable for all $\equiv_i \in I$.

The inductive claim is that there is a choice for Q_α , i.e., a countable and dense subset of M_α that shares properties (1-3) of N_α stated above. For then, $T \restriction \alpha$ can be extended by a level T_α of nodes corresponding to the branches in Q_α , in a way that guarantees, that the relevant t.e.r.s and tree homomorphisms extend to the new level.

First choose a left-inverse mapping $\sigma_{x,d}$ of $\overline{\rho}_{x,d}$ for all pairs $(x, d) \in F_\alpha$. In order to construct Q_α pick a countable and dense subset Z_α of N_α . Extend Z_α in countably many steps to close it under the application of the mappings

$$\overline{\varphi}_{x,d} \text{ and } \overline{\varphi}_{x,d}^{-1} \text{ for } (x, d) \in E_\alpha$$

and

$$\overline{\rho}_{x,d} \text{ and } \sigma_{x,d} \text{ for } (x, d) \in F_\alpha$$

while at the same time rendering this set suitable for all t.e.r.s under consideration. Finally choose a bijection $j : Q_\alpha \rightarrow \omega(\alpha + 1) \setminus \omega_\alpha$ and extend $<_T$ in the obvious way to define T_α .

7.4. Extend \equiv_i for successor i . Next we define \equiv_i on T_α in the case, that $i = j + 1$. (Recall that for limit $i = i(\alpha)$ the relation \equiv_i on T_α is determined by its behaviour below.) Letting α^- be the maximum of $\alpha \cap C_i$, $t \in T_\alpha$ and $r := t \restriction \alpha^-$, we easily see that we have an infinite set $\hat{r} \cap t / \equiv_j$ (here we view T_α as a dense subset of $[T \restriction \alpha]$) which we partition into \aleph_0 infinite sets $P(t / \equiv_j, r, n)$, each of them dense in $\hat{r} \cap t / \equiv_j$. Note, that the definition of P has to be independent from the choice of t . Finally we define \equiv_i on T_α by letting $s \equiv_i t$ if and only if

$$s \restriction \alpha^- \equiv_i t \restriction \alpha^- \text{ and } (\exists n \in \omega) s \in P(s / \equiv_j, s \restriction \alpha^-, n) \text{ and } t \in P(t / \equiv_j, t \restriction \alpha^-, n).$$

Then by construction, \equiv_i is ∞ -nice as well as $\equiv_{j/i}$.

7.5. Check isomorphisms. To finish the induction step, we verify that the extension procedure for the tree isomorphisms succeeds. We only check the most complicated case, that of $\psi_{x,d}$ for $(x, d) \in F_\alpha$. We start and show that our choice of T_α admits extensions of the tree isomorphisms $\psi_{x,d}$ for all $(x, d) \in F_\alpha$. Fix $(x, d) \in F_\alpha$ and set $c' = (c(x) \cap d) \cup \{\alpha\}$. First of all extend \equiv_x to T_α by letting $s \equiv_x t$ if and only if $s \restriction \gamma \equiv_x t \restriction \gamma$ for all $\gamma \in c(x)$. (This is the only t.e.r. extending \equiv_x on T_α because of the normality requirement in the definition of t.e.r.)

We can identify T_α with Q_α via the bijection j . This in mind, we show that the unique extension ψ of $\psi_{x,d}$ to $(T \restriction c') / \equiv_x$ is an isomorphism onto $T \restriction c'$: for $s \in T_\alpha$

define

$$\psi'(s/\equiv_x) := \text{the unique } t \in T_\alpha \text{ with } \psi_{x,d}(s \upharpoonright \gamma / \equiv) = t \upharpoonright \gamma \text{ for all } \gamma \in c' \setminus \{\alpha\}$$

and let $\psi = \psi_{x,d} \cup \psi'$. We check the soundness of this definition. For $s \in T_\alpha$ there is $\psi(s/\equiv_x) = j \circ \bar{\rho}_{x,d} \circ j^{-1}(s)$. The normality of T/\equiv guarantees that the definition of $\psi(s/\equiv_x)$ is independent from the choice of the representative s . So it is clear that ψ is indeed a tree homomorphism.

Now, by the definition of \equiv_x on T_α , if $s \neq t$ for $s, t \in T_\alpha$, then there is some $\gamma \in c' \setminus \{\alpha\}$ with $s \upharpoonright \gamma \not\equiv_x t \upharpoonright \gamma$ and so $\psi(s/\equiv_x) \neq \psi(t/\equiv_x)$. On the other hand, for every $s \in T_\alpha$ we have

$$\psi(j \circ \sigma_{x,d} \circ j^{-1}(s)/\equiv_x) = s.$$

So ψ is bijective also on the top level of $(T \upharpoonright c')/\equiv$ and therefore a tree isomorphism.

For reference in later induction steps we denote this tree isomorphism ψ by $\psi_{x,d}$. This completes the inductive construction.

7.6. Chain homogeneity. It remains to show that the above construction yields a Souslin tree T whose regular open algebra \mathbb{B} is chain homogeneous. We omit the standard argument proving that T is Souslin.

If we are given two generating chains K, K' of \mathbb{B} let $X \subset \omega_1$ be a code with respect to g for a pair of wipers on the club $C \subset \{\alpha : \omega\alpha = \alpha\}$ representing K and K' . Let D be a club set in ω_1 associated to X by \diamond^+ . Then for each $\alpha \in E = \{\gamma \in C \cap D \mid \gamma = \bigcup(\gamma \cap C \cap D)\}$ the construction of the tree gives us a tree automorphism $\varphi_{X \cap \alpha, D \cap \alpha}$ of $T \upharpoonright (C \cap D \cap \alpha)$, and the union of that increasing chain,

$$\varphi = \bigcup_{\alpha \in E} \varphi_{X \cap \alpha, D \cap \alpha},$$

extends to an automorphism φ of \mathbb{B} that carries K to K' .

Now let \mathbb{A} be a complete and atomless subalgebra of \mathbb{B} represented by the decent t.e.r. \equiv on $T \upharpoonright C$. Let D be a club, such that the \diamond^+ -sequence guesses $D \cap \alpha$ and $X \cap \alpha$ for a code X for \equiv for all $\alpha \in D$. Then the construction yields a tree isomorphism $\psi : (T \upharpoonright C \cap D)/\equiv \rightarrow T \upharpoonright C \cap D$ and therefore a Boolean isomorphism between \mathbb{B} and \mathbb{A} .

So let K be a maximal chain of \mathbb{B} , $\mathbb{A} = \langle K \rangle^{\text{cm}}$ and $\psi : \mathbb{B} \cong \mathbb{A}$. Now K is isomorphic to the generating chain $\psi^{-1} K$ and is thus of that unique order type.

7.7. Verification of the properties of the ∞ -nice chain. Let i be a countable limit ordinal. Then, by our definition of \equiv_i (in the limit case), for every node $s \in T \upharpoonright C_i$, its \equiv_i -class is just the intersection over the family $(s/\equiv_j)_{j < i}$. This shows that $\mathbb{A}_i = \langle \bigcup_{j < i} \mathbb{A}_j \rangle^{\text{cm}}$, i.e. continuity of the chain.

For the proof of (b.ii) let $(\mathbb{C}_\nu \mid \nu < \lambda)$ be given. By chain homogeneity of \mathbb{B} we can choose $\varphi_0 : \mathbb{A}_0 \cong \mathbb{C}_0$, and then inductively extend the given $\varphi_\nu : \mathbb{A}_\nu \rightarrow \mathbb{C}_\nu$ to $\varphi_{\nu+1} :$

$\mathbb{A}_{\nu+1} \rightarrow \mathbb{C}_{\nu+1}$ by virtue of condition (a) as follows. Choose an isomorphism $\varphi_{\nu+1}^0 : \mathbb{A}_{\nu+1} \rightarrow \mathbb{C}_{\nu+1}$ and extend $\psi' := (\varphi_{\nu+1}^0)^{-1} \circ \varphi_\nu$ by virtue of (a) to an automorphism ψ of $\mathbb{A}_{\nu+1}$. Then set $\varphi_{\nu+1} := \varphi_{\nu+1}^0 \circ \psi$.

At limit stage μ , the argument given above shows that

$$\left\langle \bigcup_{\nu < \mu} \mathbb{C}_\nu \right\rangle^{\text{cm}} = \mathbb{C}_\mu = \left\langle \bigcup_{\nu < \mu} \varphi_\nu'' \mathbb{A}_\nu \right\rangle^{\text{cm}}.$$

So there is a unique extension φ of $\bigcup_{\nu < \mu} \varphi_\nu$ with domain $\mathbb{A}_\mu = \langle \bigcup_{\nu < \mu} \mathbb{A}_\nu u \rangle^{\text{cm}}$ and range $\mathbb{C}_\mu = \langle \bigcup_{\nu < \mu} \mathbb{C}_\nu \rangle^{\text{cm}}$.

This finishes the proof of Theorem 5.2. \square

8. FEATURES, REMARKS AND OPEN PROBLEMS

8.1. Features of chain homogeneous Souslin algebras. It is not hard to see that by omission of the additional steps (for properties a) and b) of Theorem 5.2) in the construction we arrive at a small Souslin algebra which is plainly chain homogeneous. The following two propositions state properties shared by all chain homogeneous Souslin algebras.

Proposition 8.1. *A small and chain homogeneous Souslin algebra \mathbb{B} is homogeneous in the following strong sense. For every pair $\mathbb{A}_0, \mathbb{A}_1$ of Souslin subalgebras and $x \in \mathbb{A}_0$ and $y \in \mathbb{A}_1$ where $0 <_{\mathbb{B}} x, y <_{\mathbb{B}} 1$, there are 2^{\aleph_1} distinct isomorphisms $\varphi : \mathbb{A}_0 \rightarrow \mathbb{A}_1$ with $\varphi(x) = y$.*

Proof. We first argue that \mathbb{B} is weakly homogeneous. \mathbb{B} is assumed to be small. So for every pair a, b of non-zero elements of \mathbb{B} there are generating chains K, K' of \mathbb{B} with $a \in K$ and $b \in K'$. Each isomorphism φ between K and K' satisfies $\varphi(a) \cdot b \neq 0$ and extends to an automorphism of \mathbb{B} .

By a theorem proved independently by Koppelberg and by Solovay (cf. [ŠR89, Thm.18.4.1]), every complete and weakly homogeneous Boolean algebra is a power of a homogeneous factor, which in our case is isomorphic to \mathbb{B} , because of the c.c.c. satisfied by \mathbb{B} .

As there are 2^{\aleph_1} distinct generating chains, chain homogeneity implies that \mathbb{B} has 2^{\aleph_1} automorphisms. The rest is then routine. \square

Proposition 8.2. *a) If \mathbb{A} is an atomless, complete subalgebra of the Souslin algebra \mathbb{B} , and \mathbb{A} and \mathbb{B} are isomorphic, then no atomless, complete subalgebra of \mathbb{B} can be independent from \mathbb{A} .
b) A chain homogeneous Souslin algebra has no independent pair of atomless and complete subalgebras.*

Proof. The proof of a) is straight forward applying the fact that a Souslin algebra cannot have an independent pair of mutually isomorphic Souslin subalgebras. For the proof of b) note, that here we have not assumed \mathbb{B} to be small. In the case of a small Souslin algebra, an application of part a) suffices to prove b). But also a big Souslin algebra \mathbb{B} has only maximal chains, that completely generate subalgebras which are small Souslin algebras. So assume that $\mathbb{A}_0, \mathbb{A}_1$ form an independent pair of complete and atomless subalgebras of \mathbb{B} . Then there are maximal chains $K_0 \subset \mathbb{A}_0$ and $K_1 \subset \mathbb{A}_1$, which are isomorphic to each other. The isomorphism between the chains extends to an isomorphism between the two subalgebras that are completely generated by the chains:

$$\mathbb{C}_0 := \langle K_0 \rangle^{\text{cm}} \subset \mathbb{A}_0 \text{ and } \mathbb{C}_1 := \langle K_1 \rangle^{\text{cm}}.$$

But then again, we have an isomorphic pair of subalgebras $\mathbb{C}_0, \mathbb{C}_1$ of \mathbb{B} that cannot be independent unless \mathbb{B} fails to satisfy the countable chain condition. \square

8.2. The number of non-isomorphic maximal chains. Classical questions about Souslin trees often circulated around the number isomorphism types or the number of automorphisms. In this respect, it is worth noting, that a Souslin algebra which is not chain homogeneous has at least 2^{\aleph_0} non-isomorphic maximal chains and that the regular open algebras of most Souslin trees considered in set theoretic literature, such as strongly homogeneous or free Souslin trees have 2^{\aleph_1} non-isomorphic maximal (even generating) chains.

8.3. On the hypothesis. Concerning the hypothesis (\diamond^+) met for our constructions, we remark the following. In [AS93, §6] and [AS85, §4] a model of $\text{ZFC} + \neg\text{SH}$ is constructed, in which there is no homogeneous Souslin tree. Yet a (chain) homogeneous Souslin algebra always has a homogeneous Souslinization. So we have found a ZFC-model where Souslin's hypothesis fails, but which has no chain homogeneous Souslin algebras.

It is open whether we could make do with strictly less than \diamond^+ , e.g., if the assumption of \diamond is sufficient to guarantee the existence of a chain homogeneous Souslin algebra, cf. the open problems section below.

However, we can show that the existence of chain homogeneous Souslin algebras with properties as strong as stated in Theorem 5.2 cannot be obtained from \diamond alone.

Proposition 8.3. *Let \mathbb{B}^* be the big and chain homogeneous algebra of Theorem 5.3. Forcing with \mathbb{B}^* turns any ground model Souslin tree T , which is regularly embeddable into \mathbb{B}^* , into a Kurepa tree.*

Proof. Let $G \subset \mathbb{B}_{\omega_2}$ be a V -generic filter, and let $\mathbb{A} := \langle T \rangle^{\text{cm}}$. For $\alpha < \omega_2$ fix in V an isomorphism $\varphi_\alpha : \mathbb{B}_\alpha \cong \mathbb{A}$ and in $V[G]$ define $G_\alpha := \mathbb{B}_\alpha \cap G$.

Staying in $V[G]$, for $\alpha < \omega_2$, the set

$$b_\alpha := \{t \in T \mid (\exists g \in G_\alpha) \varphi_\alpha(g) \leq_\mathbb{A} t\}$$

is an ω_1 -branch of T , and we have $V[G_\alpha] = V[b_\alpha]$ for all $\alpha < \omega_2$. Finally, if $\gamma < \beta < \omega_2$ then $b_\beta \in V[b_\beta] \setminus V[b_\gamma]$, because the map φ_γ is in V . In particular we have $b_\beta \neq b_\gamma$. \square

Since the Iteration Theorem 5.3 only assumes the strong properties of the Souslin algebra of Theorem 5.2 we have the following result on the hypotheses used.

Corollary 8.4. *Assume that there is a Mahlo cardinal. Then there is a model of $\text{ZFC} + \diamond$ in which there is no Souslin algebra with the properties stated in theorem 5.2.*

Proof. In [Jen] Jensen considered the generic Kurepa hypothesis $\text{GKH} :=$ “there is a c.c.c. forcing that forces KH in the generic extension”. He shows that if κ is a Mahlo cardinal then GKH is false in the Levy-style generic extension collapsing κ to become \aleph_2 . This partial order always forces \diamond (cf. [Kun80, Exercises VIII.J.5/6]). Finally, \mathbb{B}_{ω_2} is c.c.c. and forces, by the last Proposition, KH . \square

As far as we know, the proofs of Theorems 5.2 and 5.3 give the first construction of a big Souslin algebra assuming the principle \diamond^+ only. Jensen’s constructions use \diamond and \square , cf. [DJ74] or [Jen]. In [Jec73, §5] gives a forcing that adjoins a so-called Souslin mess. This is a partial order of partial functions generalising the notion of a Souslin tree. The regular open algebra of a large enough Souslin mess is a big Souslin algebra. Laver has constructed a Souslin mess, only using \diamond and Silver’s principle W . (W is a strengthening of Kurepa’s Hypothesis KH ; cf. [Jec97, (24.16)].)

Nevertheless, the following remains open.

Question 8.5. Assuming \diamond , is there a chain homogeneous Souslin algebra?

This is only a sample question, as \diamond could be replaced by any of its variants known to be strictly weaker than \diamond^+ .

In [DJ74, §V.3], a Souslin tree with at least \aleph_2 automorphisms (a property in common with chain homogeneous Souslin algebras) is constructed under \diamond^+ , and on p. 51 the authors remark: *It is doubtful whether this is provable from \diamond* . Though it is not hard to see that the regular open algebra of this tree is homogeneous for generating chains, we do not know whether it is fully chain homogeneous.

As argued above, it is impossible to carry out our construction of Section 7 under \diamond alone.

8.4. Souslin’s hypothesis minus one.

Question 8.6. Is there a model of ZFC with exactly one Souslin line (up to isomorphism)?

Such a property could be paraphrased as “Souslin’s hypothesis minus one” as there would be only one counter-example, essentially, i.e. only one without separable intervals. In such a model the associated Boolean algebra would have to be chain homogeneous. Anyway, also if we refocus the question on Souslin trees or Souslin algebras, then the respective algebra must admit isomorphisms to all of its Souslin subalgebras, a feature in common with the chain homogeneous Souslin algebras.

In an attempt to construct such a model by variations of the well-known iterated forcing techniques of Solovay-Tennenbaum (variation for a strongly homogeneous Souslin tree by Larson, cf. [Lar99, §4]) or that of Jensen (variation for a free Souslin tree by Abraham, cf. [AS85, §4]), which leave one a priori specified Souslin tree intact, the main obstacle is to ensure that enough of the homogeneity properties of the preserved Souslin algebra remain valid in the final model.

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